

SEMATECH

*1996 Applied Reliability Tools
Workshop (ARTWORK IX)*

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Computer Simulation of Repairable Processes

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Introduction

Computer simulation

- ◆ Powerful tool to recreate possible outcomes of random processes
- ◆ Allows checking models and analysis procedures without extensive experimentation
- ◆ Where theory is limited, may be the only avenue for solving complex problems
- ◆ Source of data for training

Outline of Topics

- ◆ Discuss a general procedure for simulating data for any distribution
- ◆ Apply to example for estimating parameters of Weibull distribution
- ◆ Review concepts of repairable systems
 - Renewal processes
 - Non-renewal processes
- ◆ Describe simulation procedure for non-renewal model
- ◆ Illustrate using power relation model for NHPP

Overview

- ◆ Illustrate the value of computer simulation for solving many different types of simple and complex problems in design, manufacturing, analysis, etc.
- ◆ Show the power of EXCEL for simulating, analyzing, and modeling random processes

Simulating a Distribution

Generating random events for a specified situation.

Want to **produce failure times** from a **specific distribution**.

To simulate **any distribution**, we will use a special property of the cumulative distribution function (**CDF**).

Simulating Random Variables

Let U be a uniform random variable in the interval $(0, 1)$.

*If $F(t)$ is any CDF with an inverse $t = F^{-1}$, then substituting U for F in the inverse expression generates a **random variable T** distributed according to F .*

Weibull Distribution

Example

The CDF for a Weibull distribution with characteristic life ***c*** and shape parameter ***m*** is

$$F(t) = 1 - e^{-(t/c)^m}$$

F(t) is the **fraction failing** in the population by time ***t***.

F(t) is also the probability of failure by time ***t***.

The inverse expression is :

$$t = F^{-1} = c[-\ln(1 - F)]^{1/m}$$

Simulating Weibull Times to Failure

Substitute the unit uniform random variable U in the inverse expression to get :

$$T = F^{-1}(U) = c[-\ln(1 - U)]^{1/m}$$

Since both $1-U$ and U are a uniformly distributed variate in the interval $(0,1)$, simplify the inverse expression as :

$$T = c[-\ln U]^{1/m}$$

Each U value produces a time to failure T from a Weibull distribution with parameters c and m .

Weibull Simulation

Example in EXCEL

Estimating the parameters ***c*** and ***m*** of a Weibull distribution is not simple, even for complete data. For MLE's, iterative numerical method are required.

Let's investigate the suitability of the following simple procedure for estimating the parameters.

Write the 25th and 75th percentiles for a Weibull distribution as:

$$t_{25} = c[-\ln(1 - 0.25)]^{1/m} = c[\ln(4 / 3)]^{1/m}$$

$$t_{75} = c[-\ln(1 - 0.75)]^{1/m} = c[\ln(4)]^{1/m}$$

Parameter Estimation for Weibull Distribution

Solve the second equation for **c** and substitute into the first to get

$$t_{25} = t_{75} \frac{[\ln(4/3)]^{1/m}}{[\ln(4)]^{1/m}} = [0.207519]^{1/m}$$

or

$$m \ln\left(\frac{t_{25}}{t_{75}}\right) = \ln(0.207519) = -1.572534$$

Solve for **m** directly in terms of the ratio of the two percentiles.

$$\hat{m} = \frac{-1.572534}{\ln\left(\frac{t_{25}}{t_{75}}\right)}$$

Parameter Estimation for Weibull Distribution

With m , we can easily solve for c using any convenient percentile such as the median, that is,

$$\hat{c} = \frac{t_{50}}{[\ln(2)]^{1/\hat{m}}}$$

How well does this procedure work for fitting a Weibull model to data?
To get an answer, let's try simulation in EXCEL.

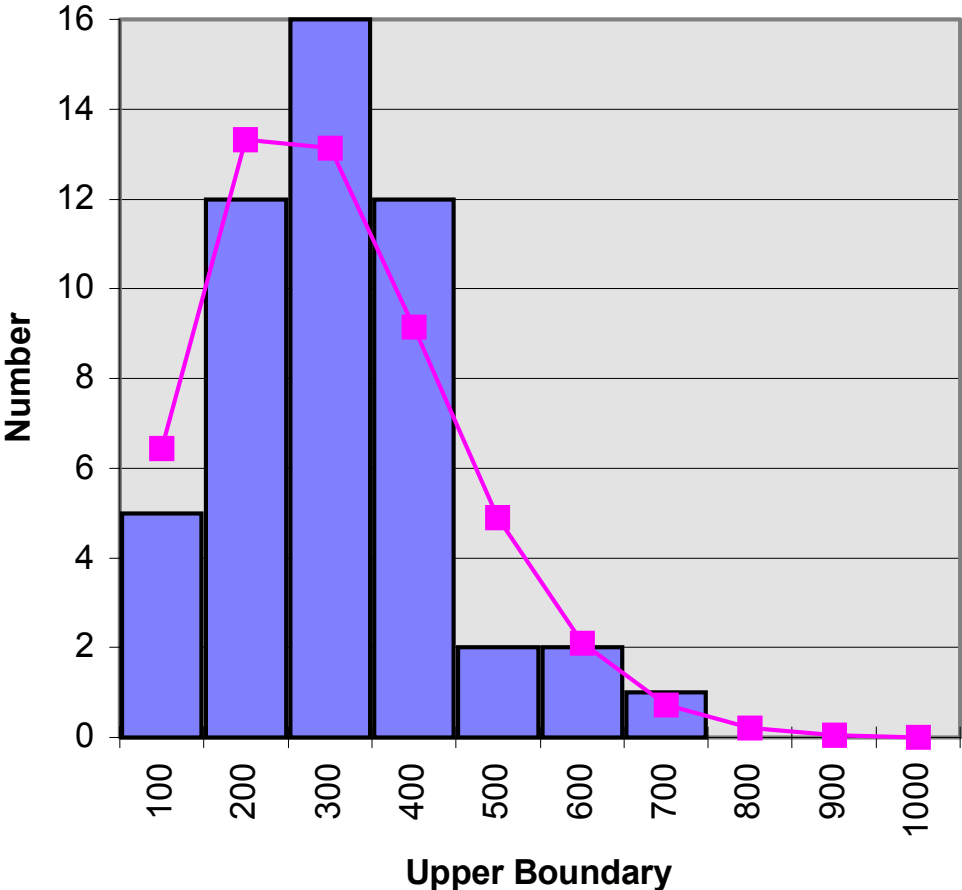
Weibull Distribution Simulation in EXCEL

We generate 50 pseudo-random numbers in a column using the EXCEL function **=rand()**. The adjacent column, based on the inverse CDF calculation, contains the random times distributed according to a Weibull with specified parameters ***c*** and ***m***.

After estimating the parameters according to the described procedure, we produce a histogram of the data along with a model Weibull fit.

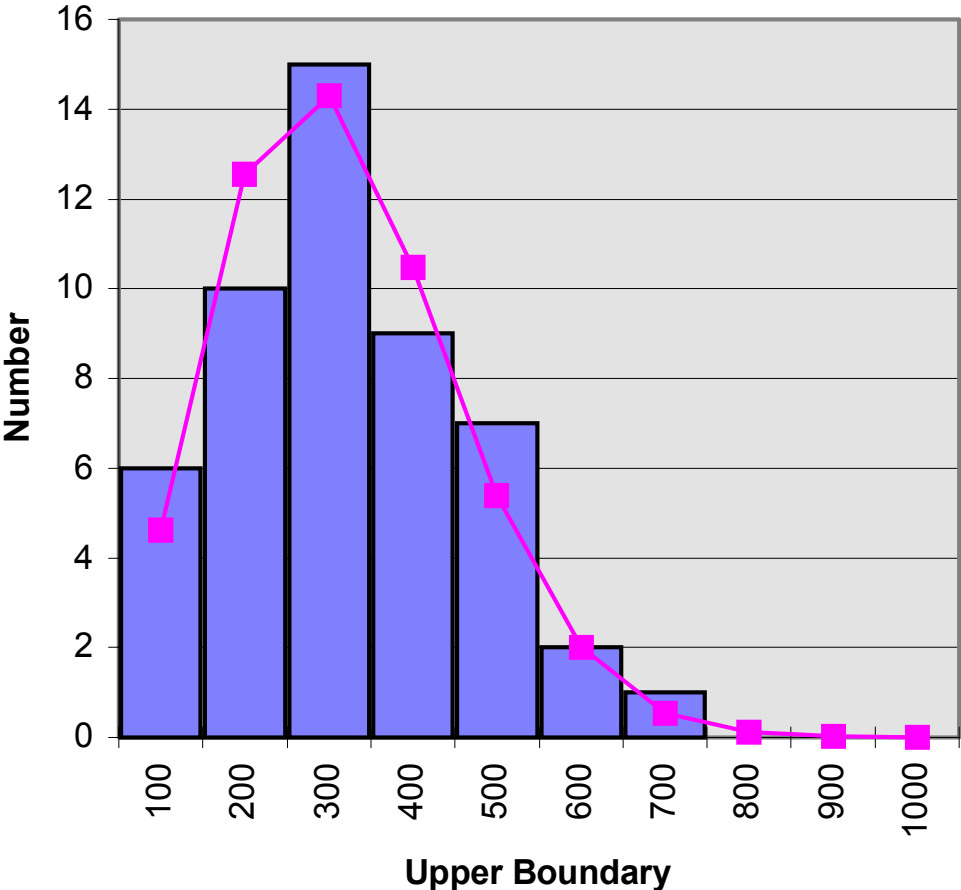
Simulation Results After One Run

Simulation Results



Simulation Results After Second Run

Simulation Results



Adding Simulation Runs

By making additional simulation runs, and collecting results from each simulation, we can determine many facets of the problem, such as:

the sampling distribution of the estimates

- ◆ confidence intervals on the estimates
- ◆ the suitability of the Weibull model based on these estimates

Simulation for Repairable Systems

- ◆ Need to generate random failure events associated with repairable processes.
- ◆ Let's first develop some definitions and concepts for repairable systems.

Concepts of Repairable Systems

□ A system is ***repairable*** if it can be restored to satisfactory operation by any action, including replacement of components, changes to adjustable settings, swapping of parts, or even a sharp blow with a hammer.

Renewal Process

- ◆ Special case of repairable system
- ◆ Times between failures are independent and identically distributed (i.i.d.), that is, the times come from a single population
- ◆ No trend

Analysis of a Renewal Process

- ◆ Reliability analysis methods for non-repairable components have applicability
- ◆ Times between failures may be combined into a single group for analysis

Homogeneous Poisson Renewal Process

If the times between repairs, X_i , are i.i.d. from an exponential distribution with PDF

$$f(x) = \lambda e^{-\lambda x}$$

having constant failure rate λ , then, the number of repairs $N(t)$ by system age t , has a Poisson distribution with mean $M(t) = \lambda t$.

Thus, the probability of observing exactly $N(t) = k$ failures in the interval $(0, t)$ is

$$P[N(t) = k] = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

Mean Cumulative Repair Function

- ◆ The mean cumulative repair function, $M(t)$, is the average or expected number of repairs for $N(t)$ in time t .
- ◆ For a HPP, the rate of repairs has a constant value equal to λ .

Simulating a Renewal Process

- ◆ Specify the distribution
- ◆ Generate the random variables from the distribution via simulation
- ◆ Arrange the failure times sequentially
- ◆ The consecutive, running sums of the failure times represent the system age at each failure point

Simulating a Renewal Process: NHPP Example

- ◆ Assume we have a NHPP with the mean time between repairs (MTBF) of 100 hours.
- ◆ Let's see what random outcomes are possible for 20 failures in the life of a system.
- ◆ To simulate this process, we substitute twenty unit uniform random numbers, ***U***, in the expression:

$$T = -100 \times \ln U$$

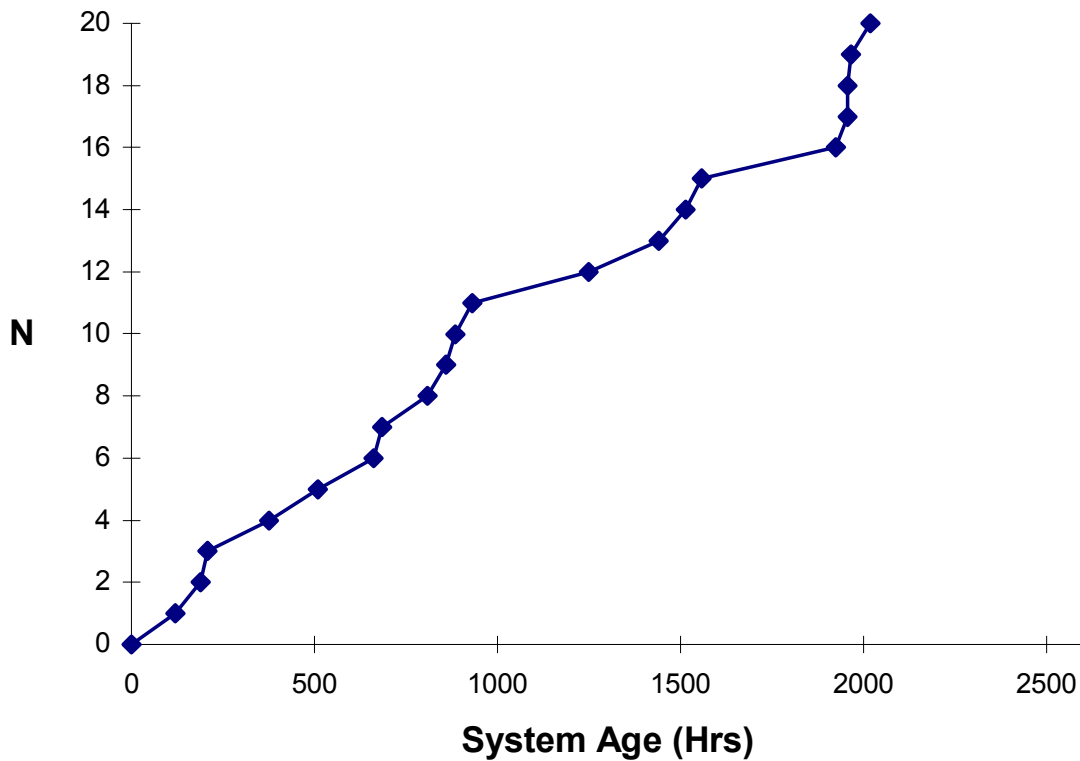
Simulation of HPP in EXCEL

Use =rand() for U .

U	Xi	System Age T	No. Repairs
		0	0
0.300	120.3	120.3	1
0.505	68.4	188.7	2
0.821	19.7	208.4	3
0.187	167.5	375.9	4
0.265	132.7	508.7	5
0.214	153.9	662.6	6
0.807	21.4	684.1	7
0.288	124.5	808.6	8
0.601	50.9	859.5	9
0.775	25.5	885.0	10
0.632	46.0	931.0	11
0.041	319.1	1250.1	12
0.150	189.6	1439.7	13
0.471	75.4	1515.1	14
0.650	43.1	1558.2	15
0.026	366.1	1924.3	16
0.723	32.5	1956.7	17
0.990	1.0	1957.7	18
0.929	7.4	1965.1	19
0.580	54.5	2019.6	20
MTBF	101.0		

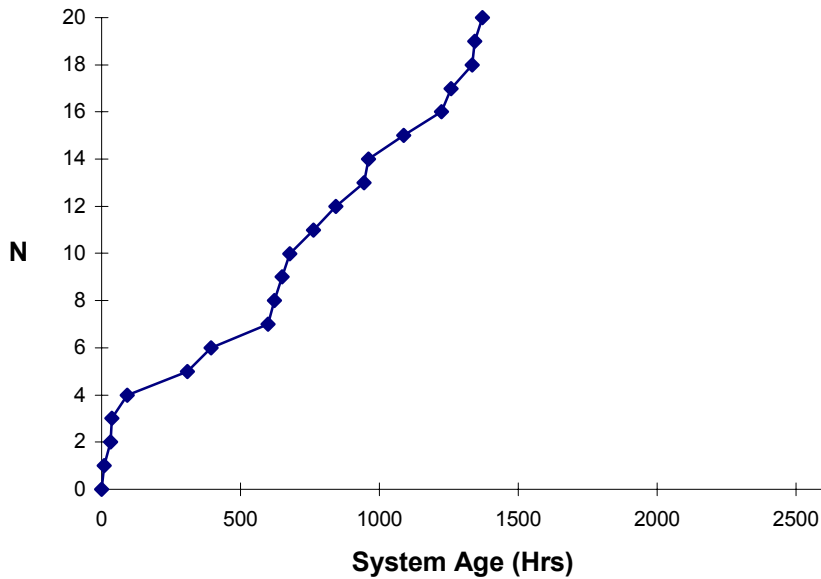
Plot of One HPP Simulation Run

Cumulative Repairs vs System Age

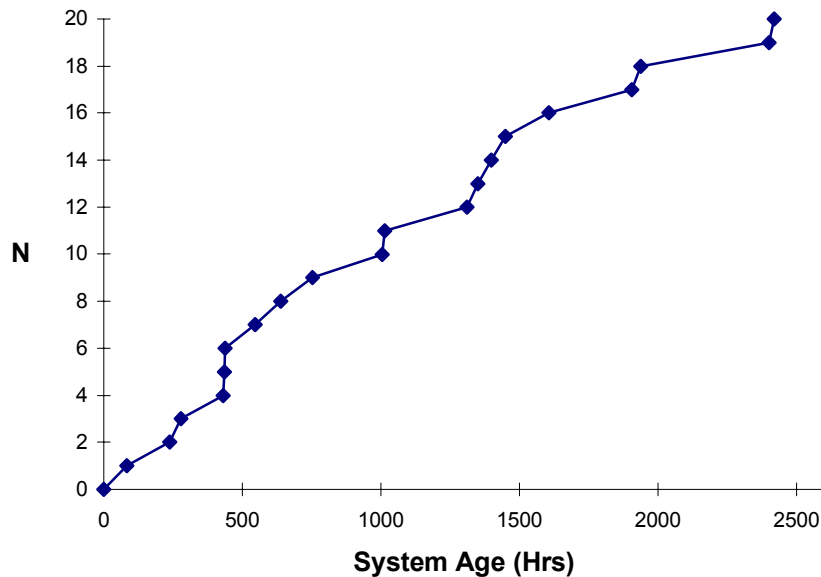


Two More HPP Simulation Runs (MTBF = 100)

Cumulative Repairs vs System Age



Cumulative Repairs vs System Age



Non-Renewal Repair Process

- ◆ The times between failures are not from a single distribution
- ◆ The times between failures may not be independent
- ◆ There may be trend present, that is, the times between repairs may be growing (system reliability getting better) or decreasing (system reliability getting worse).

Modeling a Non-Renewal Repair Process

- ◆ We need to specify a model for simulation.
- ◆ We can compare model results to any observed repair history.
- ◆ A popular model is a generalization of the HPP. Called the non-homogeneous Poisson process and denoted NHPP, the rate of repairs or intensity λ becomes a function $\lambda(\mathbf{t})$ of system age \mathbf{t} .

NHPP Model

The question for a NHPP is what form should the intensity function $\lambda(t)$ [and consequently $M(t)$] take?

Consider a NHPP with mean repair function modeled by a **power relation**

$$M(t) = at^b .$$

The intensity function is

$$\lambda(t) = \frac{dM(t)}{dt} = abt^{b-1}$$

Note for $b > 1$, the repair rate is increasing in time (system degrading). For $b < 1$, the repair rate decreases (system improvement).

Caution on Terminology

Although the previous expression looks like the hazard rate for a Weibull distribution, leading some to refer to this NHPP as a Weibull process, there is **no relation** to a Weibull distribution.

Simulating a Power Relation NHPP

- ◆ General approach presented in Ross, ***Introduction to Probability Models***.
- ◆ Formulas for NHPP with intensity function described by power relation derived in Tobias and Trindade, ***Applied Reliability, 2nd ed.***

Simulating a NHPP with Power Relation Intensity

Formulas

System age to first repair

$$Y_1 = \left[-\frac{1}{a} \ln U_1 \right]^{\frac{1}{b}}$$

System age to second repair

$$Y_2 = \left[Y_1^b - \frac{1}{a} \ln U_2 \right]^{\frac{1}{b}}$$

System age to i th repair

$$Y_i = \left[Y_{i-1}^b - \frac{1}{a} \ln U_i \right]^{\frac{1}{b}}$$

Simulation Studies

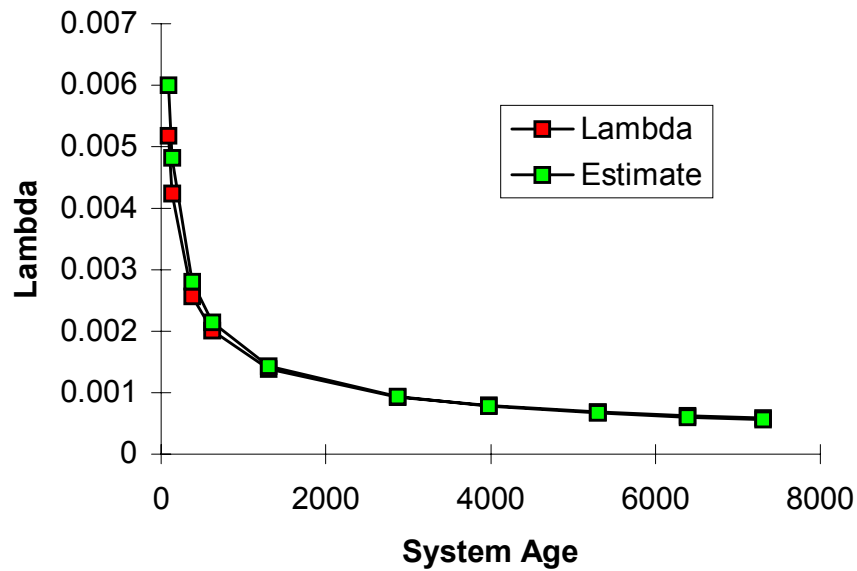
- ◆ Check adequacy and variation in parameter estimation formulas
- ◆ Plot confidence intervals along with model estimates
- ◆ Try different representations of displaying results
- ◆ Experiment with new estimation formulas
- ◆ Reveal weaknesses in understanding NHPP processes
- ◆ Show range of variability in possible system behavior

Simulation of Power Relation NHPP in EXCEL

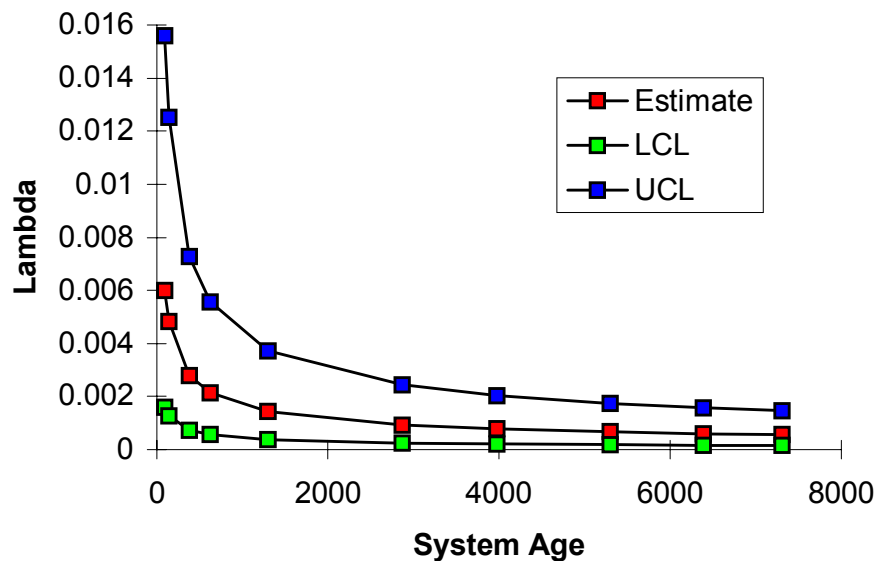
Simulation of NHPP with intensity $ab t^{(b-1)}$				MLE Estimates (Failure Truncate		
a	0.1			0.1530		
b	0.5			0.4580		
Power Relation Model					Model	Model
	U	t	N	t in v	M(t)	M(t) Est
X0	0.305861	0	0	0	0.00	0.00
X1	0.380877	93	1	100	0.97	1.22
X2	0.806257	139	2	400	1.18	1.47
X3	0.463942	380	3	900	1.95	2.32
X4	0.581419	620	4	1600	2.49	2.91
X5	0.327763	1301	5	2500	3.61	4.08
X6	0.173829	2869	6	3600	5.36	5.87
X7	0.386943	3976	7	4900	6.31	6.81
X8	0.376467	5303	8	6400	7.28	7.77
X9	0.490924	6390	9	8100	7.99	8.47
X10	0.575409	7304	10	10000	8.55	9.00

Example of Power Relation NHPP Simulation Runs

Intensity Function



MLE Lambda and CI



Summary

- ◆ Various ways of simulating data for specific distributions, renewal processes, and non-homogeneous Poisson processes
- ◆ Examples of applying simulation techniques to understand theory and check approaches to solving problems
- ◆ Power of EXCEL to facilitate simulation and analysis

Where to get more information

- ◆ Tobias, P. A. and Trindade, D. C., ***Applied Reliability***, 2nd ed., (1995), Van Nostrand Reinhold
- ◆ Ross, S., ***A Course in Simulation***, (1990), Macmillan